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SOME PRIME ELEMENTS IN THE LATTICE OF INTERPRETABILITY TYPES

BY

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ABSTRACT. A general theorem is proved which implies that the types of PA (Peano Arithmetic), ZF (Zermelo-Fraenkel Set Theory) and GB (Gödel-Bernays Set Theory) are prime in the lattice of interpretability types.

0. Introduction. One of the important goals of mathematical logic is to investigate the strength of theories. A good approximation to the intuitive concept of strength is the quasiordering "T is interpretable in S". We shall consider interpretation in a very general sense which was introduced by Mycielski in [**5**]. Mycielski has shown (among other things) that after canonical factorization of that quasiordering a distributive lattice is obtained. The elements of the lattice will be called types; thus every theory determines a type—the type to which it belongs.

An important task is to determine the prime elements of the lattice. (An element is *prime* or *join-irreducible* if it is not the join of two smaller elements.) Following Mycielski we shall call a theory *connected* if its type is prime. We introduce the concept of sequential theory; roughly speaking a theory is sequential iff it permits some coding of any finite sequence of elements. The main theorem of this paper (Theorem 4.2) says that every sequential theory is connected. This answers a question of [**1**] since PA, ZF, GB and $\mathrm{Th}(\omega; +, \cdot)$ are sequential theories. (Actually we arrived at the concept of sequential theories when trying to generalize a former proof of connectedness for these theories.)

In §§2 and 3 we prove some lemmas and theorems about sequential theories which are prerequisites for the proof of the main theorem. Though we prove only a little bit more general statements than we need for the main theorem, they already show that interesting mathematics can be developed in every sequential theory. §4 is devoted to the rest of the proof of the main theorem. We use the sufficient condition for being connected found by Mycielski in [**1**]. Related problems are discussed in the last section. Using our theorem we shall show that if equality is treated as a logical symbol then the lattices of interpretability types are different from Mycielski's lattice and that a concept of sequentiality introduced by Vaught [**10**] cannot be used for generalizing our theorem.

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1. Preliminaries. In this paper we consider the first order logic without equality and without function symbols. A theory is an arbitrary set of sentences. For a structure $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A})$ denotes its complete theory. We shall consider also theories which contain equality; in such theories we shall use function symbols in order to abbreviate complicated formulae.

By an *interpretation* we understand *multidimensional local interpretation with parameters*. Assuming that the reader knows what an interpretation is, we shall explain only the adjectives. *Multidimensionality* means that variables are translated to d-tuples of variables for some fixed d (thus e.g. d-dimensional geometry can be interpreted in the theory of real closed fields). However, except for the last section, all the interpretations that we shall need will be one-dimensional; this is because of Theorem 1.1, which reduces the problem to one-dimensional interpretations. T is *locally* interpretable in S iff every finite set of axioms of T is interpretable in S; T is *globally* interpretable in S if a single translation works for all axioms at once. A sentence φ is *interpretable with parameters* in T if there is a translation ψ of φ with some free variables (called parameters) such that T proves the existential closure of ψ; (thus e.g. $\mathrm{Th}(\omega; \leqslant)$ can be interpreted in $\mathrm{Th}(Z; \leqslant)$, where Z is the set of integers and ω is the set of nonnegative integers). It is more convenient to get rid of free variables in the interpretations by using constants. If T proves $\exists x_1, \ldots, x_n \alpha(x_1, \ldots, x_n)$ then we allow to introduce new constants $c_1, \ldots, c_n$ and add the axiom $\alpha(c_1, \ldots, c_n)$ to T. Of course, φ is interpretable in T with parameters iff φ is interpretable without parameters in an extension by constants of T. *Therefore in the whole paper we shall work with formulae with constants and "T proves φ" or "$T \vdash \varphi$" will always mean that φ is provable in an extension of T by constants.* Finally, recall that equality need not be translated as equality since it is considered as a nonlogical symbol. (For a precise definition of the concept of interpretability see **[5]**.)

Let $\delta(x)$ be a formula with one free variable. An interpretation is said to be *relativized to the domain* $\delta(x)$ if the quanitifers $\forall x, \exists x$ are translated by $\forall x(\delta(x) \Rightarrow \cdots)$, $\exists x(\delta(x) \,\&\, \cdots)$, respectively. An interpretation is unrelativized if the quantifiers are translated identically. When defining the lattice of interpretability types we do not have to consider relativized interpretations, since every such interpretation can be transformed into an unrelativized one by way of "blowing up an element so that the whole universe is filled up". Relativized interpretations will be important in the proof of the main theorem (see Lemma 3.4 and Theorem 3.5).

We shall use the following convention in order to make formulae shorter and more comprehensible. Instead of writing $\forall x(\delta(x) \Rightarrow \cdots)$ we shall write $(\forall x \in \delta)(\cdots)$ and similarly for the existential quantifier. This alone would not help much, thus we shall also fuse several quantifiers of the same type and with the same bound; e.g. we write $(\forall x, y \in \delta)(\cdots)$ instead of $(\forall x \in \delta)(\forall y \in \delta)(\cdots)$. We shall also use bounded quantifiers of the form $(\forall x \leqslant y)$ and $(\exists x \leqslant y)$, etc.

The *disjoint product* $\mathfrak{A} \dot{\times} \mathfrak{B}$ of structures $\mathfrak{A}$ and $\mathfrak{B}$ is the structure whose support is the product $A \times B$ of the supports of $\mathfrak{A}$ and $\mathfrak{B}$ and whose relations are cylinders over the relations of $\mathfrak{A}$ and $\mathfrak{B}$. E.g. if R is a binary relation in $\mathfrak{A}$ then there is a corresponding binary relation R_A in $\mathfrak{A} \dot{\times} \mathfrak{B}$ defined by

$$(a, b) R_A (a', b') \Leftrightarrow_{\mathrm{df}} a R a',$$

while if R were a relation in $\mathfrak{B}$ we would have

$$(a, b)R_B(a', b') \Leftrightarrow_{\mathrm{df}} bRb'.$$

In order to show that the theory of linear orderings without maximal elements is connected, Mycielski used the following characterization [**1**].

THEOREM 1.1. *A theory T is connected if and only if, for every two structures $\mathfrak{A}$ and $\mathfrak{B}$ and every finite fragment T' of T if T' has a one-dimensional interpretation in* $\mathrm{Th}(\mathfrak{A} \dot{\times} \mathfrak{B})$, *then T' has an interpetation in* $\mathrm{Th}(\mathfrak{A})$ *or in* $\mathrm{Th}(\mathfrak{B})$. □

(The part "only if" is not in [**1**], but it is almost trivial.) The following lemma has also been proved in [**1**].

LEMMA 1.2. *If R is a relation definable in $\mathfrak{A} \dot{\times} \mathfrak{B}$ then its projection to A (resp. to B) is definable in $\mathfrak{A}$ (resp. in $\mathfrak{B}$).* □

An improved version of [**1**], containing proofs of 1.1 and 1.2, is being prepared for publication.

2. Sequential theories.

DEFINITION 2.1. A theory T is called *sequential* if it contains the equality, along with the axioms of equality for all predicates of T, and one can find formulae $N(x)$, $x \leqslant y$ and $\beta(x, y, z)$ such that the following are provable in T:

(2.1) $\leqslant$ is a linear ordering of N and each element of N has a successor in N;

(2.2) $\forall x, y (\forall n \in N) \exists z \forall t (\forall m \leqslant n)$

$$[\beta(t, m, z) \Leftrightarrow (\beta(t, m, x) \,\&\, m < n) \vee (t = y \,\&\, m = n)].$$

(We write $m < n$ instead of $m \leqslant n \,\&\, m \neq n$.)

The intended meaning of $\beta(t, n, x)$ is: t is the nth element of the sequence encoded by x. Then (2.2) says, roughly speaking, that each codable sequence can be extended by an arbitrary element. The elements of N will be called *numbers* (and denoted usually by m, n); the *successor* of n will be denoted by $n + 1$.

It is well known that PA, ZF and GB are sequential. E.g. in GB one defines $\beta(X, n, Y)$ by

$$\forall x (x \in X \Leftrightarrow (x, n) \in Y).$$

We shall show that a large part of the number theory can be developed in every sequential theory.

Sequential theories can also be defined as follows. First, let the *elementary sequential theory* be the theory with three relation symbols N, $\leqslant$, β (unary, binary, and ternary, resp.) and with equality, axiomatized by the axioms of equality, (2.1) and (2.2). Then a theory T with equality is sequential iff there is an unrelativized one-dimensional interpretation of the elementary sequential theory in T such that the interpretation of equality is absolute. Thus, if we worked in logic with equality, then any extension of a sequential theory (by new predicates and axioms) would be sequential too. Thus e.g. $\mathrm{Th}(\omega; +, \cdot)$ is sequential since PA is sequential. Observe that the natural interpretation of ZF in GB *is* relativized, therefore the sequentiality of GB does not follow from the sequentiality of ZF.

LEMMA 2.2. *Let T be a sequential theory. Then we can find formulae $N(x)$, $x \leqslant y$, $\beta(x, y, z)$ satisfying the definition of sequential theories and, moreover,*

(2.3) $(N; \leqslant)$ *has a smallest element and the other elements of N have predecessors*;

(2.4) $\exists x \forall t, n \neg\beta(t, n, x)$;

(2.5) $\forall t_1, t_2, n, x\, (\beta(t_1, n, x) \,\&\, \beta(t_2, n, x) \Rightarrow t_1 = t_2)$;

(2.6) $\forall x (\exists k \in N) \forall n [\exists t \beta(t, n, x) \Leftrightarrow n < k]$.

PROOF. Let T be sequential and let $N(x)$, $x \leqslant y$, $\beta(x, y, z)$ be some formulae satisfying (2.1) and (2.2). First we define new $N(x)$, say $N^*(x)$, by

$$N(x) \,\&\, 0 \leqslant x \,\&\, \forall y (0 < y \leqslant x \Rightarrow y \text{ has the predecessor}).$$

Here 0 is a constant for which we assume $N(0)$. The new β is defined by

$$\beta^*(t, n, x) \Leftrightarrow_{\text{df}} \beta(t, n+1, x) \,\&\, (\exists k \in N^*)\big[\beta(k, 0, x) \,\&\, n < k \,\&\, (\forall m \in N^*) \\ (m < k \Rightarrow \exists! t \beta(t, m, x))\big].$$

The new ordering is the restriction of $\leqslant$ to N^*. □

Further, the *smallest element* of N will always be denoted by 0.

In this paper an important role will be played by the definable initial segments of N closed under successors. They will be called cuts. More precisely, let T be a sequential theory, $I(x)$ a formula. Then we call I a *cut* (in T) if T proves

$$I(0) \,\&\, (\forall n \in I)\big[I(n+1) \,\&\, N(n) \,\&\, (\forall m \leqslant n) I(m)\big].$$

We shall say that a cut J is *below* I or write $J \subseteq I$ if $T \vdash \forall n\, (J(n) \Rightarrow I(n))$. We are going to prove analogs of some classical theorems. A common feature of them will be that the classical prefix $(\forall n \in N)$ will be replaced by $(\forall n \in I)$ for some cut I, which will be constructed for each particular theorem. If the prefix $(\forall n \in I)$ is followed by a formula which does not contain N, we can replace N by I and (2.1)–(2.6) will remain true. Thus we will show that we can assume more about N, $\leqslant$, β in the definition of the sequential theories.

Standing assumption. From now till Theorem 2.7 we work in a sequential theory T, and assume N, $\leqslant$, β satisfy (at least) (2.1)–(2.6). However, instead of β we shall use a more intuitive notation. Namely, we define a partial function $x, n \mapsto x[n]$ by

$$x[n] = t \Leftrightarrow_{\text{df}} \beta(t, n, x).$$

We shall use this partial function in terms also; in such cases we shall always assume that $x[n]$ is defined for every x, n in the range of quantification.

LEMMA 2.3 (RECURSION). *Let $f(x, n, m)$, $g_0(x)$ be functions definable in T. (We use number variables n, m only to indicate that we do not care about the values of f for other individuals.)*

(a) *Then it is possible to define in T a cut J and a function $g(x, n)$ such that*

(2.7) $T \vdash \forall x\big[g(x,0) = g_0(x) \,\&\, (\forall n \in J)(g(x, n+1) = f(x, n, g(x, n)))\big]$;

(b) *and if, moreover, for a formula $\sigma(x, y)$,*

$$T \vdash \forall x, y, z(\sigma(x, y) \,\&\, \sigma(y, z) \Rightarrow \sigma(x, z)) \,\&\, \forall x (\forall m, n \in J)(\sigma(m, f(x, n, m))),$$

then

$$T \vdash \forall x (\forall n, m \in J)(n < m \Rightarrow \sigma(g(x, n), g(x, m))).$$

PROOF. (a) Let $A(a, m, x)$ be a formula expressing that a encodes a function defined on the segment $[0, m]$ of N and satisfying property (2.7) for all n in $[0, m]$, i.e.

$$(\forall n \leqslant m)\exists t(a[n] = t) \,\&\, a[0] = g_0(x) \,\&\, (\forall n < m)(a[n+1] = f(x, n, a[n])).$$

Define $B(b, m, x)$ by $\exists a(A(a, m, x) \,\&\, b = a[m])$. It is still possible that more than one b satisfies this formula for given m, x. Therefore we define $I(m, x)$ by

$$N(m) \,\&\, (\forall n \leqslant m)\exists! b B(b, n, x).$$

It follows from (2.1)–(2.6) that $I(m, x)$ is a cut for every x, hence also $J(m) \Leftrightarrow_{\mathrm{df}} \forall x I(m, x)$ is a cut. Define g by

$$\begin{aligned} g(x, m) &= b \quad \text{if } B(b, m, x) \,\&\, J(m), \\ &= 0 \quad \text{if there is no such } b. \end{aligned}$$

Then g satisfies (2.7).

(b) One can add the following formula to the definition of $J(m)$:

$$\forall n, l, m, b, c(n < l \leqslant m \,\&\, B(b, n, x) \,\&\, B(c, l, x) \Rightarrow \sigma(b, c)),$$

and use the proof above. It is easy to verify that J is a cut again. □

LEMMA 2.4. *It is possible to define a cut I in T and three operations $m + n$, $m \cdot n$, 2^n such that T proves*

$$(2.8) \qquad (\forall m, n \in I)(m + 0 = m \,\&\, m + (n+1) = (m+n) + 1 \\ \&\, m \cdot 0 = 0 \,\&\, m \cdot (n+1) = m \cdot n + m);$$

$$(2.9) \qquad (\forall n \in I)(2^0 = 0 + 1 \,\&\, 2^{n+1} = 2^n + 2^n).$$

PROOF. Set

$$\begin{aligned} g_0(x) &= x, \\ f(x, n, m) &= m + 1 \quad \text{if } N(m), \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned}$$

Let $m + n = g(m, n)$ and J_1 be the function and the cut given by Lemma 2.3. In order to define $\cdot$, set

$$g_0(x) = 0, \qquad f(x, n, m) = m + x.$$

Using Lemma 2.3, we get an operation $\cdot$ and a cut J_2. In a similar fashion we obtain 2^n and a cut J_3. Then we define $I(n)$ by $J_1(n) \,\&\, J_2(n) \,\&\, J_3(n)$. □

We work here with total functions, thus the operations above are defined (in some unnatural way) for nonnumbers also, and it is possible that $m + n$ etc. is not a number for some numbers m, n.

Before defining a cut that is closed under $+$ and $\cdot$ we need one more lemma.

LEMMA 2.5 (INDUCTION). *For every formula $\varphi(n, x)$ one can define a cut J such that T proves*

$$(2.10) \quad \forall x[\varphi(0, x) \,\&\, (\forall n \in J)(\varphi(n, x) \Rightarrow \varphi(n+1, x)) \Rightarrow (\forall n \in J)\varphi(n, x)].$$

PROOF. Define $J(n)$ by

$$N(n) \,\&\, \forall x[\varphi(0,x) \,\&\, (\forall m < n)(\varphi(m,x) \Rightarrow \varphi(m+1,x)) \Rightarrow (\forall m \leqslant n)\varphi(m,x)].$$

Then (2.10) follows easily. □

Consider, for example, associativity of $+$. Then let $\varphi(n, m, l)$ be $(m+l)+n = m+(l+n)$. Then, assuming (2.8.),

$$T \vdash (\forall n, m, l \in I)(\varphi(0, m, l) \,\&\, (\varphi(n, m, l) \Rightarrow \varphi(n+1, m, l))).$$

By Lemma 2.5, there is a cut J such that

$$T \vdash (\forall n \in J)(\forall m, l \in I)((m+l)+n = m+(l+n)).$$

(We can use two parameters m, l instead of one x because of coding.) Thus $+$ is associative on J. In the same way we can find cuts on which $\cdot$ is associative, and $+, \cdot, 2^n$ are nondecreasing. Taking the intersection of these cuts we get a cut on which all these properties are satisfied.

LEMMA 2.6. *Suppose that the operations $m+n$, $m \cdot n$, and a cut I are defined in T such that T proves* (2.8). *Then it is possible to define a cut K such that T proves $K \subseteq I$ and*

$$(\forall m, n \in K)(K(m+n) \,\&\, K(m \cdot n)),$$

i.e. K is closed under $+$ and $\cdot$.

PROOF. By the considerations above we can assume that $+$ and $\cdot$ are associative and nondecreasing. We define

$$K_1(n) \Leftrightarrow_{\mathrm{df}} (\forall m \in I) I(n+m).$$

Then if $K_1(n)$, $K_1(m)$, and $I(l)$, we have $I(m+l)$, hence $I(n+(m+l))$. Thus, by associativity, $I((n+m)+l)$. This proves that K_1 is closed under $+$. In the same way one can define, from K_1, a cut K closed under $\cdot$, hence closed under both operations. □

R. Solovay was probably the first to observe that, given a nonempty initial segment closed under successors in a model of PA, one can define initial segments closed under addition, multiplication, $n^{\lfloor \log_2 n \rfloor}$, $n^{\lfloor \log_2 n \rfloor^{\lfloor \log_2 \log_2 n \rfloor}}$, etc. Paris and Dimitracopoulos [6] proved that in some initial segments closed under successors it is not possible to define an initial segment closed under all these operations simultaneously. Since 2^n grows faster than all these operations, one cannot define initial segments closed under 2^n in such segments. The theory of a model of PA with an initial segment is sequential, since PA is sequential. This gives us an example of a cut in a sequential theory in which we cannot define a cut closed under 2^n.

A *bounded arithmetical formula*, or Δ_0 *formula*, is a formula in the language 0, 1, $+$ and $\cdot$ and in which only quantifiers of the form $(\exists x \leqslant n)$ and $(\forall y \leqslant n)$ occur, where n is a fixed variable, henceforth called the *bound*. The scheme of *bounded induction*, or $I\Delta_0$, is the set of sentences of the form

$$(2.11)\quad \forall x_1,\ldots,x_k[\varphi(0, x_1,\ldots,x_k) \,\&\, \forall n(\varphi(n, x_1,\ldots,x_k) \Rightarrow \varphi(n+1, x_1,\ldots,x_k)) \Rightarrow \forall n \varphi(n, x_1,\ldots,x_k)],$$

where φ is Δ_0 with the bound n. Given a finite subset of $I\Delta_0$, it is easy to find a cut I such that I is closed under $+$ and $\cdot$ and the relativizations of the formulae of this subset to I are provable in T: just apply Lemmas 2.5 and 2.6 several times. Thus $I\Delta_0$ is locally interpretable in T. (We cannot use Lemma 2.5 to show that induction for unbounded arithmetical formulae is interpretable in T; if we start, say, with a formula $\varphi(n) = \forall m \psi(n, m)$ we cannot relativize the quanitifer $\forall m$ to the cut J.)

The same thing can be done if we include the exponentiation in the language except that we have to treat it as a relation rather than a function. In fact, it is not even necessary to do that. By a result of J. H. Bennett, exponentiation can be defined by a bounded formula and the basic properties of it are provable from (finitely many instances of) $I\Delta_0$ (cf. [**2**]).

In the proof of the main theorem we can manage with finitely many instances of $I\Delta_0$. We need uniqueness of the diadic expansions of numbers and a couple of other properties; all that can be very easily checked to follow from (a finite part of) $I\Delta_0$. However, more can be proved. Let Q be the theory with equality, a constant 0, a unary operation S, two binary operations $+$ and $\cdot$, and the axioms of which are the universal closures of the following formulae:

$$S(x) = S(y) \Rightarrow x = y; \quad 0 \neq S(y); \quad x \neq 0 \Rightarrow \exists y(x = S(y));$$
$$x + 0 = x; \qquad x + S(y) = S(x + y);$$
$$x \cdot 0 = 0; \qquad x \cdot S(y) = x \cdot y + x$$

(and the axioms of equality) cf. [**9**].

THEOREM 2.7 (DUE TO A. WILKIE). *There exists a formula $J(x)$ such that* 0, $S(0)$, $+$, $\cdot$, *and domain $J(x)$ determine a global interpretation of Peano Arithmetic with induction only for bounded formulae in* Q.

We shall sketch the idea of a proof only briefly.

Since we do not have an ordering in Q, we use, instead of cuts, formulae $I(x)$ satisfying only

$$\mathrm{Q} \vdash I(0) \,\&\, (\forall x \in I) I(S(x)).$$

One can easily prove Lemmas 2.5 and 2.6 with this modification. Thus one can find a domain on which sufficiently many instances of $I\Delta_0$ hold, where the ordering is defined by

$$m \leqslant n \Leftrightarrow_{\mathrm{df}} \exists p(p + m = n).$$

On such a domain one can define coding of sequences, relations etc.

The next step consists in finding a formula $\Theta(m, n, x)$ which defines the truth of Δ_0 formulae on some cut I; i.e. for every $\varphi \in \Delta_0$ with bound n there is a numeral t such that Q proves

$$(\forall n, x_1, \ldots, x_k \in I)(\varphi(n, x_1, \ldots, x_k) \Leftrightarrow \Theta(t, n, \langle x_1, \ldots, x_k \rangle)),$$

where $\langle \cdots \rangle$ is a suitable coding of sequences of numbers by numbers. These are infinitely many conditions for Θ, but we do not need infinitely many instances of $I\Delta_0$ for proving them. We can take Θ so that Tarski-type conditions for Θ are satisfied (such as "$\varphi_1 \& \varphi_2$ is true iff φ_1 is true and φ_2 is true") and then all the conditions above are assured. Roughly speaking, one can take $\Theta(t, n, \langle x_1, \ldots, x_k \rangle)$ to be a formula expressing that the formula with the number t is true iff there exists a number z which encodes a satisfaction relation for formulae with the numbers $0, 1, \ldots, t$ and parameters smaller or equal to $\max(n, x_1, \ldots, x_k)$. Such a z would be too large for some n and might not exist, therefore we have to restrict ourselves to a cut I. (More details can be found in [7].)

Now we define (in a similar manner as in the proof of Lemma 2.5) $J(n)$ by

$$I(n) \& (\forall t, x \in I)[\Theta(t, 0, x) \& (\forall m < n)$$
$$(\Theta(t, m, x) \Rightarrow \Theta(t, m+1, x)) \Rightarrow (\forall m \leqslant n)\Theta(t, m, x)].$$

Assuming good behaviour of Θ on I we can prove that J is a cut. If J is not closed under $+$ and $\cdot$, we can take a subcut of J which is closed under these operations (by Lemma 2.6). □

It is an open problem whether Peano Arithmetic with induction only for Δ_0 formulae is finitely axiomatizable. If it were, then the theorem above would follow easily from the fact that $I\Delta_0$ is locally interpretable in T.

Also, the question of how large a fragment of Peano Arithmetic can be interpreted in every sequential theory is open. We know that there are theorems of PA which do not follow from $I\Delta_0$ and can be interpreted in every sequential theory together with $I\Delta_0$. There are theorems which can be interpreted but for which it is not known whether they follow from $I\Delta_0$; e.g. the theorem "there are infinitely many primes".

By way of ending this section we sum up the most important facts about the numbers in a sequential theory (now the term "numbers" being justified).

COROLLARY 2.8. *A theory T with equality is sequential if and only if it is possible to find formulae $N(x)$, $x \leqslant y$, $\beta(x, y, z)$, and define operations $n + m$, $n \cdot m$, 2^n such that T proves*

(a) *conditions* (2.1)–(2.6.);

(b) *N is closed under $+$ and $\cdot$*;

(c)

$$(\forall m, n \in N)(m + 0 = m \& m + (n+1) = (m+n) + 1$$
$$\& m \cdot 0 = 0 \& m \cdot (n+1) = m \cdot n + m);$$

(d) $\forall x(N(2^x) \Rightarrow 2^{x+1} = 2^x + 2^x) \& 2^0 = 0 + 1$;

(e) $(N; 0, +, \cdot, 2^n)$ *is an interpretation of bounded induction for formulae including partial function 2^n also.*

(*In* (c), (d), $a + 1$ *denotes the successor of* a.)

PROOF. Using Lemmas 2.2, 2.4, 2.6, it is easy to find formulae $N_1(x)$, $x \leqslant y$, $\beta(x, y, z)$, $x + y$, $x \cdot y$ which satisfy (a), (b), (c) for a given sequential theory T. Then N_1, $x + 1$, $+$, $\cdot$ determine an interpretation of Q in T. Hence by Theorem 2.7 we can find a cut $N(x)$ on which $I\Delta_0$ is satisfied. By a result mentioned above, we can define exponentiation satisfying (d). Instead of using the technically involved result about definability of exponentiation, we can define exponentiation using Lemma 2.4 and check that Theorem 2.7 can be proved with Q enriched by the relation "$m = 2^n$".

3. Interpretations and trees in sequential theories. In this section we prove two important results about sequential theories (Theorem 3.5 and Lemma 3.6). We work in a sequential theory T and we assume for convenience that we have the relations and operations satisfying Corollary 2.8. Using diadic expansion, we can consider numbers also as sequences of 0's and 1's. We define the *length* of n by

$$|n| \text{ is the smallest } r \text{ such that } n < 2^r,$$

and *concatenation* of two numbers m, n by

$$m \circ n = m \cdot 2^{|n|} + n.$$

LEMMA 3.1. *Given $l \in \omega$, it is possible to define functions $f_1(a_1)$, $f_2(a_1, a_2), \ldots, f_l(a_1, \ldots, a_l)$ such that, for every cut J, T proves*

(3.1) *if J is closed under $\cdot$, then J is closed under $f_1, \ldots, f_l$;*

(3.2) *for every $i, j = 1, \ldots, l$, $i < j$, $a_1, \ldots, a_j \in N$,*

$$\max(a_1, \ldots, a_i) < f_i(a_1, \ldots, a_i) < f_j(a_1, \ldots, a_j);$$

(3.3) *for every $n \in N$ there is at most one $i \leqslant l$ and one i-tuple $a_1, \ldots, a_i$ such that $n = f_i(a_1, \ldots, a_i)$.*

PROOF. Define for $i = 1, \ldots, l$,

$$f_i(a_1, \ldots, a_i) = (a_1 + 1) \circ (a_2 + 1) \circ \cdots \circ (a_i + 1)$$
$$\circ \cdots \circ (a_i + 1) \circ 2^{|a_1|} \circ 2^{|a_2|} \circ \cdots \circ 2^{|a_i|} \circ 2^i.$$

Since $m \circ n \leqslant m \cdot 2 \cdot n + n$, the functions f_i, $i = 1, \ldots, l$, are bounded by polynomials of $a_1, \ldots, a_i$, which proves (3.1). Conditions (3.2), (3.3) follow easily from $I\Delta_0$. □

Lemma 3.1 will serve to represent terms in the proof of Lemma 3.4.

Our next interest is to prove a weak version of the scheme of choice. The natural form of the scheme of choice for sequential theories, i.e.

> "for every $\varphi(n, x)$ it is possible to define a cut $J(n)$ and a function $f(n)$ such that $T \vdash (\forall n \in N)\exists x \varphi(n, x) \Rightarrow (\forall n \in J)\varphi(n, f(n))$",

is not provable e.g. for ZF without the axiom of choice. It turns out that if, for every n, there are not "too many" numbers x satisfying $\varphi(n, x)$, then choice is possible.

LEMMA 3.2 (CHOICE). (a) *Suppose that, for some cut K, formula $\varphi(n, m)$ and $k \in \omega$, T proves $(\forall n \in K)(\exists m \leqslant 2^{n^k + 1})\varphi(n, m)$. Then it is possible to define a cut $J \subseteq K$ and a function $f(n)$ in T such that T proves*

(3.4) $(\forall n \in J)\varphi(n, f(n))$.

(b) *Suppose that, for some cut K, formulae $\varphi(n, m)$, $\sigma(n, m)$ and $k \in \omega$, T proves*

(3.5) $\exists m \varphi(0, m)$;

$$(3.6)\quad (\forall n, m, l \in K)\Big[(\sigma(m, n) \,\&\, \sigma(n, l) \Rightarrow \sigma(m, l))$$
$$\&\, \big(\varphi(n, m) \Rightarrow (\exists m' \leqslant 2^{n^k+1})(\varphi(n+1, m') \,\&\, \sigma(m, m'))\big)\Big].$$

Then it is possible to define a cut $J \subseteq K$ and a function $f(n)$ in T such that T proves (3.4) *and*

$$(\forall n \in J)(\forall m < n)\sigma(f(m), f(n)).$$

PROOF. (a) Define

$$(3.7)\qquad \begin{aligned} f(n) &= \text{"the smallest } m \text{ such that } \varphi(n, m)\text{"},\\ f(n) &= 0 \text{ if there is no smallest } m \text{ such that } \varphi(n, m). \end{aligned}$$

Let $I_1(m) \Leftrightarrow_{\mathrm{df}} (\forall n \leq m)\varphi(n, f(n))\ \&\ N(m)$. If I_1 is a cut, we are done. Otherwise it has the largest element. Let n be its successor and let

$$I_2(m) \Leftrightarrow_{\mathrm{df}} (\forall l \leq m)\neg\varphi(n, l)\ \&\ N(m).$$

Then I_2 is a cut and does not contain 2^{n^k+1}. Let I_3 be some cut below I_2 closed under $+$, and let I_4 be the cut defined by $I_4(m) \Leftrightarrow_{\mathrm{df}} I_3(2^m)$. Finally, let I be a subcut of I_3 closed under $\cdot$. The existence of such cuts follows from Lemma 2.6. Since I_2 does not contain 2^{n^k+1}, I_4 does not contain $n^k + 1$. Hence I does not contain n. But n is the first element for which $\varphi(n, f(n))$ fails. Therefore I satisfies (3.4).

(b) It is not difficult to obtain a proof of (b) combining the proofs of (a) and the proof of Lemma 2.3(b). Thus we leave it to the reader. □

The bound 2^{n^k+1} of Lemma 3.2 can be replaced by 2^{2^n}, $2^{2^{2^n}}$, etc.; we conjecture that the lemma is not provable with larger bounds.

The next lemma is a formalization of the following easy theorem of set theory: Let A_n, be a set of k-ary relations on $[0, n)$, for $n \in \omega$, such that $A_n \neq \varnothing$, and every relation of A_n can be extended to a relation of A_{n+1}, for $n \in \omega$; then there is a k-ary relation C on ω such that, for every $n \in \omega$, the restriction of C to $[0, n)$ belongs to A_n.

LEMMA 3.3. *Let $\alpha(n, x)$, $\rho(x, c_1, \ldots, c_k)$, $I(n)$ be formulae such that T proves that I is a cut and*

(3.8) $\exists x\alpha(0, x)$;

$$(3.9)\quad (\forall n \in I)\forall x\exists y\big[\alpha(n, x) \Rightarrow \alpha(n + 1, y)\ \&\ (\forall c_1, \ldots, c_k < n)\big(\rho(x, c_1, \ldots, c_k) \Leftrightarrow \rho(y, c_1, \ldots, c_k)\big)\big].$$

Then there are formulae $\gamma(c_1, \ldots, c_k)$, $J(n)$, such that T proves that J is a subcut of I and

$$(3.10)\quad (\forall n \in J)\exists x\big[\alpha(n, x)\ \&\ (\forall c_1, \ldots, c_k < n)\big(\rho(x, c_1, \ldots, c_k) \Leftrightarrow \gamma(c_1, \ldots, c_k)\big)\big].$$

PROOF. Let α, ρ, I be given. We can imagine that, given n in I, every x encodes a k-ary relation on $[0, n)$, namely the relation consisting of k-tuples $c_1, \ldots, c_k$ such that $\rho(x, c_1, \ldots, c_k)$ holds. First we have to encode these relations by numbers. Thus we define

$$(3.11)\quad \begin{aligned} &\text{"}p \text{ is a code of } x \text{ on } n\text{"} \Leftrightarrow_{\mathrm{df}} p < 2^{n^k}\ \&\ (\forall c_1, \ldots, c_k < n)\\ &\big(\text{the } \big(c_1 + c_2 \cdot n + \cdots + c_k \cdot n^{k-1}\big)\text{th digit of } p \text{ is } 1 \text{ iff } \rho(x, c_1, \ldots, c_k)\big). \end{aligned}$$

We shall show that there is a cut K such that, for every n in K, every x has a code. Let I_1 consist of all m's such that for every $n \leq m$ and every x, there is a code of x on n. If I_1 does not have the largest element, we are done. Otherwise, let n be the successor of the largest element of I_1. Then some x does not have a code on n.

Define $I_2(m)$ by

(3.12) $(\exists p < 2^m)(\forall c_1,\ldots,c_k < n)\big(c_1 + c_2 \cdot n + \cdots + c_k \cdot n^{k-1} \leqslant m \Rightarrow (3.11)\big).$

Suppose $I_2(m)$, and let p be determined by (3.12). We want to find some q in order to satisfy (3.12) also for $m + 1$. One can easily derive from $I\Delta_0$ that there exists at most one k-tuple $c_1,\ldots,c_k$ such that

$$c_1 + c_2 \cdot n + \cdots + c_k \cdot n^{k-1} = m + 1.$$

If there is such a k-tuple and, moreover, $\rho(x, c_1,\ldots,c_k)$ holds, then put $q = 2^m + p$. Otherwise let $q = p$. Thus the $(m + 1)$th digit of q is 1 iff $\rho(x, c_1,\ldots,c_k)$ holds. The other properties required for a cut can also be easily verified for I_2. Let K be a cut below I_2 which is closed under $\cdot$. Since I_2 does not contain n^k, K does not contain n. Therefore every x has a code on m, for every m in K.

Let p be a code of x on n. In order to be able to apply Lemma 3.2 we have to include information about n into the code. Therefore we shall use $2^{n^k} + p$ instead of p. Define $\varphi(n, m)$ by

(3.13) $(\exists x, p)\big(\alpha(n, x)\ \&\ p \text{ is a code of } x \text{ on } n\ \&\ m = 2^{n^k} + p\big),$

and, for $p < 2^{n^k}$, $q < 2^{m^k}$, define $\sigma(2^{n^k} + p, 2^{m^k} + q)$ by

(3.14) $n \leqslant m\ \&\ (\forall c_1,\ldots,c_k < n)$
$\big(\text{the } (c_1 + c_2 \cdot n + \cdots + c_k \cdot n^{k-1})\text{th digit}$
$\text{of } p \text{ is } 1 \Leftrightarrow \text{the } (c_1 + c_2 \cdot m + \cdots + c_k \cdot m^{k-1})\text{th digit of } q \text{ is } 1\big).$

That is, $\varphi(n, m)$ expresses that m encodes a relation satisfying α on $[0, n)$, and $\sigma(n, m)$ expresses that the relation encoded by m is a prolongation of the relation encoded by n. By (3.8) we have (3.5). The transitivity of σ follows from the definition of σ. The rest of the condition (3.6) follows from (3.9). Let f be the function given by Lemma 3.2. Define $\gamma(c_1,\ldots,c_k)$ by

(3.15) $(\exists m, q)\big(f(m) = 2^{m^k} + q\ \&\ c_1 < n\ \&\cdots\&\ c_k < n$
$\&\ \text{the } (c_1 + c_2 \cdot m + \cdots + c_k \cdot m^{k-1})\text{th digit of } q \text{ is } 1\big).$

We shall check condition (3.10). Let $n \in J$ be given. Let p be such that $2^{n^k} + p = f(n)$, then p is a code of some x on n such that $\alpha(n, x)$, by (3.13) and (3.5). Let $c_1,\ldots,c_k < n$ and suppose $\rho(c_1,\ldots,c_k)$. Then the $(c_1 + c_2 \cdot n + \cdots + c_k \cdot n^{k-1})$th digit of p is 1, hence $\gamma(c_1,\ldots,c_k)$. Now suppose $\gamma(c_1,\ldots,c_k)$. Let m and q of (3.15) be given and suppose $n \leqslant m$. Since $\sigma(f(n), f(m))$, the $(c_1 + c_2 \cdot n + \cdots + c_k \cdot n^{k-1})$th digit of p is 1 too. Thus $\rho(c_1,\ldots,c_k)$. The case of $m \leqslant n$ is similar. □

Lemma 3.4. *Let φ be a sentence provable in T, and let I be a cut. Then there is an interpretation of φ in T with domain I.*

Proof. Let φ be provable in T. The interpretation will be given by a Herbrand universe definable in T. We can suppose φ is of the form

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_l \exists y_l \psi(x_1, y_1, x_2, y_2,\ldots,x_l, y_l),$$

where ψ is without quanitifiers. Let $f_1(x_1),\ldots,f_l(x_1,\ldots,x_l)$, be the functions of the "algebra of terms" of Lemma 3.1. By (3.1) and Lemma 2.6, we can assume that I is closed under these functions. Let $\alpha(n, x)$ be defined by

$$(\forall m \leqslant n)\exists t(x[m] = t)$$
$$\& \bigwedge_{i=1}^{l} \forall a_1,\ldots,a_i[f_i(a_1,\ldots,a_i) \leqslant n \Rightarrow \forall x_{i+1}\exists y_{i+1}\ldots$$
$$\forall x_l \exists y_l \psi(x[a_1], x[f_1(a_1)], x[a_2], x[f_2(a_1, a_2)],\ldots,$$
$$x[a_i], x[f_i(a_1,\ldots,a_i)], x_{i+1}, y_{i+1},\ldots,x_l, y_l)].$$

The meaning of $\alpha(n, x)$ is that x encodes a sequence of length at least n, and whenever t is the $f_i(a_1,\ldots,a_i)$th element of this sequence, then t must witness the quantifier $\exists x_i$ in φ.

Claim 1. Every x such that $\alpha(n, x)$ holds can be prolonged to z such that $\alpha(n + 1, z)$ holds.

PROOF OF THE CLAIM. If $n + 1$ is not of the form $f_i(a_1,\ldots,a_i)$ for any $i = 1,\ldots,l$, then extend x arbitrarily. Suppose, now, that $n + 1 = f_i(a_1,\ldots,a_i)$. Then

$$\forall x_i \exists y_i \cdots \forall x_l \exists y_l \psi(x[a_1],x[f_1(a_1)],\ldots,x[a_{i-1}], x[f_{i-1}(a_1,\ldots,a_{i-1})], x_i, y_i,\ldots,x_l, y_l). \tag{3.16}$$

If $i = 1$, this is so because we assume that φ is provable in T. If $i > 1$, it is so because we have, by (3.2),

$$f_{i-1}(a_1,\ldots,a_{i-1}) < f_i(a_1,\ldots,a_i) = n + 1,$$

and we assume $\alpha(n, x)$. Hence, if $x[a_i]$ is substituted for x_i, there is some y that can be substituted for y_i in order to satisfy the rest of formula (3.16). Using (2.2), we can extend x to z so that $z[f_i(a_1,\ldots,a_i)] = y$. Then we have $\alpha(n + 1, z)$, because of $\alpha(n, x)$ and (3.2).

We can assume without loss of generality that φ contains only one relation symbol R (which is the only nonlogical symbol occurring in φ). Define

$$\rho(x, c_1,\ldots,c_k) \Leftrightarrow_{\text{df}} \alpha(\max(c_1,\ldots,c_k), x) \,\&\, R(x[c_1],\ldots,x[c_k]). \tag{3.17}$$

Now we apply Lemma 3.3. Condition (3.8) is trivially satisfied. Since $\alpha(n, x)$ and $m \leqslant n$ implies $\alpha(m, x)$, condition (3.9) is, in fact, Claim 1, where the last part of (3.9) is hidden in the word "prolonged". Thus we are given some $\gamma(c_1,\ldots,c_k)$ and a cut $J \subseteq I$ satisfying (3.10). As usual, we can suppose, moreover, that J is closed under $\cdot$.

Claim 2. Formula $\gamma(a_1,\ldots,a_k)$ and domain $J(a)$ determine an interpretation of φ in T.

PROOF OF THE CLAIM. Let ψ^* be the translation of ψ obtained by replacing R by γ and suitable renaming of the variables. Let $a_1,\ldots,a_l$, $b_1,\ldots,b_l < n$, and let $J(n)$ hold. Since ψ contains no quantifiers, we have, by (3.10) and (3.17), an x such that

$$\psi^*(a_1, b_1,\ldots,a_l, b_l) \Leftrightarrow \psi(x[a_1], x[b_1],\ldots,x[a_l], x[b_l]), \tag{3.18}$$

and $\alpha(n, x)$. In particular, if we consider $b_i = f_i(a_1, \ldots, a_i)$, $i = 1, \ldots, l$, then, by $\alpha(n, x)$, the right-hand side of (3.18) is satisfied; hence the left-hand side is also satisfied. Since J is closed under $\cdot$, J is closed under each of the functions $f_1, \ldots, f_l$. Thus we have shown that, whenever $a_1, \ldots, a_l$ are in J, then

$$\psi^*(a_1, f_1(a_1), \ldots, a_l, f_l(a_1, \ldots, a_l)),$$

which concludes the proof of the claim.

In order to obtain an interpretation with domain I, it is enough to blow up an element in J. □

For PA and ZF, Lemma 3.4 can be proved directly using reflexivity and the fact that there are no proper cuts in them. (This cannot be used to simplify essentially the proof of the connectivity of PA and ZF, since, when proving the connectivity of, say, ZF we need Lemma 3.4 for *extension* T of ZF in which proper cuts can be defined.) For these theories one can also prove the following.

If I is an initial segment closed under successors in a model of T, then T is globally interpretable in Th(I; $+, \cdot$).

For GB, Lemma 3.4 is implicit in a proof of R. Solovay.

Let $P(x)$ and $x \preccurlyeq y$ be formulae. We say that T proves that $(P; \preccurlyeq)$ is a *linear ordering with successors* if T proves the axioms for linear orderings relativized to P and

$$\exists x P(x) \,\&\, \forall x \in R \exists y \in P(x \preccurlyeq y \,\&\, x \neq y$$
$$\& \text{ "there is no } z \text{ strictly between } x \text{ and } y\text{")}.$$

THEOREM 3.5 ("LÖWENHEIM-SKOLEM THEOREM"). *Let T be a sequential theory; let $P(x)$, $x \preccurlyeq y$ be formulae such that T proves that $(P; \preccurlyeq)$ is a linear ordering with successors; let φ be a sentence provable in T. Then there is an interpretation of φ in T with domain P.*

PROOF. Let us choose an element s in P, and define

$$g_0(x) = s,$$
$$f(x, n, m) = \text{the successor of } m \text{ in } P \quad \text{if } P(m),$$
$$= s \quad \text{otherwise.}$$

Applying Lemma 2.3(b) to g_0, f, $\prec$ we get a cut I and a function g such that T proves

$$(\forall m, n \in I)(m < n \Rightarrow g(m) \prec g(n)),$$

(where $x \prec y$ means $x \preccurlyeq y \,\&\, x \neq y$). Thus f is a 1-1 mapping of I into P. By Lemma 3.4 there is an interpretation of φ with domain I. Using f we can shift this interpretation into P. □

The next lemma is a variant of a well-known set theoretical principle, usually refered to as König's Lemma, which says that every infinite tree has either an infinite branch or it has a node with infinitely many successors. We restrict ourselves to a special kind of tree which we shall use in the proof of the main theorem. The infinity is again represented by the domains on which it is possible to define linear orderings with successors. First we need some definitions.

We want to work with increasing sequences of numbers > 0. The most convenient way to do so is to view every number m as encoding the increasing sequence of those r's for which the rth digit of m is 1. Thus 0 represents the empty sequence and, for $m > 0$, $|m|$ is the last and the largest element of the sequence encoding by m. We define *m is an initial segment* of n by

$$m \sqsubseteq n \Leftrightarrow_{\mathrm{df}} \exists p (n = p \circ m).$$

Hence for every m, $0 \sqsubseteq m$ and every $\sqsubseteq$ successor of m is of the form $m + 2^r$, where $r \geqslant |m|$. A formula $\tau(n)$ will be called a *tree* if T proves

$$\tau(0) \,\&\, (\forall m, n \in N)(m \sqsubseteq n \,\&\, \tau(n) \Rightarrow \tau(m)).$$

The *domain* of a tree $\tau(n)$ is the formula

$$\delta_\tau(r) \Leftrightarrow_{\mathrm{df}} \exists n \in \tau(r \leqslant |n|).$$

Given a cut I, the *restriction* of τ to I is the tree defined by $\tau(n) \,\&\, I(|n|)$.

LEMMA 3.6 (KÖNIG'S LEMMA). *Let τ be a tree such that the domain of τ is a cut. Then there exists a formula $\Phi(n)$ such that T proves*

(3.19) $\forall n(\Phi(n) \Rightarrow \tau(n))$,

(3.20) $(\Phi; \leqslant)$ *is a linear ordering with successors* (*where $\leqslant$ is the ordering of the numbers*),

(3.21)

$$(\forall n, m \in \Phi)(m \leqslant n \Rightarrow m \sqsubseteq n) \vee (\exists m \in \tau)(\forall n \in \Phi)(n \textit{ is a } \sqsubseteq \textit{ successor of } m).$$

PROOF. Let τ be a tree and let a cut J be its domain. We have two orderings on τ: $\leqslant$ and $\sqsubseteq$. By saying "smaller", "larger", etc. we shall always mean $\leqslant$; the word "successor" will refer only to $\sqsubseteq$ successors in τ. If p, q are in τ, $p \sqsubseteq q \neq p$, then it is easy to prove (using bounded induction) that there is a successor s of p such that $p \sqsubseteq s \sqsubseteq q$. We cannot prove the following properties.

(3.22) If p has a successor in τ, then it has the smallest successor.

(3.23) If p and q, $p < q$, are two successors of an element in τ, then there is the smallest successor s of n in τ such that $p < s \leqslant q$. However, we can define a cut I such that, if τ is restricted to I, then τ satisfies (3.22), (3.23). Namely define $I(r)$ by

$$J(r) \,\&\, (\forall p, q < 2^r)(\text{if } p \text{ has a successor } < 2^r \text{ in } \tau, \\ \text{then it has a smallest successor } \& \text{ (3.23)}).$$

Hence we shall further assume that I is the domain of τ, and τ satisfies the conditions above. We define "n is *good*" by "the part of τ which is above n is unbounded", or more precisely by

$$\tau(n) \,\&\, (\forall r \in I)(\exists m \in \tau)(n \sqsubseteq m \,\&\, r \leqslant |m|).$$

Clearly, 0 is good.

We shall consider three cases. (Thus Φ will be a disjunction,

$$(\psi_1 \Rightarrow \Phi_1) \vee (\psi_2 \Rightarrow \Phi_2) \vee (\psi_3 \Rightarrow \Phi_3),$$

where ψ_i are the assumptions of the cases and Φ_i are the formulae constructed in the cases $i = 1, 2, 3$.)

Case 1. There exists an m in τ which is good but does not have a good successor. Let $\Phi_1(n)$ be a formula expressing that n is a successor of m and the part of τ which is above the successors of m that are smaller or equal to n is bounded. Thus (3.19) and (3.21) are satisfied. By (3.22), m has the smallest successor n_0. Since no successor of n is good, $\Phi_1(n_0)$. Suppose $\Phi_1(n)$ for some n. Since m is good and the part of τ above the successors of m smaller or equal to n is bounded, there is a successor of m larger than n. By (3.23), there is the smallest such successor, say n'. But n' is not good, thus the part of τ above n' is also bounded. Therefore $\Phi_1(n')$ holds. Thus we have (3.20).

Case 2. There exists an m in τ which is good, has a good successor, but does not have a largest good successor. Then define $\Phi_2(n)$ by "n is a successor of m and there is a good successor of m which is larger or equal to n".

Case 3. Every good m in τ has a largest good successor. Then, using Lemma 2.3(b), we can define a function g and a cut K such that

$$g(0) = 0 \,\&\, (\forall n, m \in K)[g(n+1) = \text{the largest good successor of } g(n) \,\&\, (n \leqslant m \Rightarrow g(n) \sqsubseteq g(m))].$$

Now $\Phi_3(n)$ is defined by $(\exists m \in K)(n = g(m))$. □

4. The main theorem. The last lemma that we need is of combinatorial character. The importance of the lemma will become clear in the proof of Theorem 4.2.

LEMMA 4.1. *Let a one-dimensional interpretation of a sequential theory T in $Th(\mathfrak{A} \dot{\times} \mathfrak{B})$ be given. Let N, $\leqslant$, $\cong$ be the relations that interpret numbers, ordering of numbers and the equality of T, respectively, in $\mathfrak{A} \dot{\times} \mathfrak{B}$. Then it is possible to define in $\mathfrak{A} \dot{\times} \mathfrak{B}$ a nonempty set $S \subseteq N$ such that*

(a) *every element of S has a successor in the quasiordering $(S; \leqslant)$;*

(b) *either $(a, b), (a, b') \in S \Rightarrow (a, b) \cong (a, b')$, for every a, b, b', or the same thing holds for the second coordinate.*

PROOF. Let the assumptions of the lemma be satisfied. We shall use the symbols N, $\leqslant$, $\cong$ for the formulae of T of which they are interpretations. We have to introduce some notation. Let A, B be the supports of $\mathfrak{A}$, $\mathfrak{B}$, respectively. For $x \in A \times B$ let $[x]$ be the class of the equivalence $\cong$ that contains x, i.e. $[x] = \{y \in A \times B \mid x \cong y\}$. For $X \subseteq A \times B$, let X_A be the projection of X into A, i.e. $X_A = \{a \in A \mid \exists b((a, b) \in X)\}$. X_B is defined in a similar way. Now condition (b) of the lemma can be expressed as follows:

$$(\forall x, y \in A \times B)(x \ncong y \Rightarrow ([x] \cap S)_A \cap ([y] \cap S)_A = \varnothing)$$
$$\vee\ (\forall x, y \in A \times B)(x \ncong y \Rightarrow ([x] \cap S)_B \cap ([y] \cap S)_B = \varnothing).$$

We can assume that $+$ and $\cdot$ are defined on N and $(N, \cong, \leqslant, +, \cdot)$ is a model of bounded induction. Thus we can define, for $x \in N$,

$$x^* = \{t \in N \mid \mathfrak{A} \dot{\times} \mathfrak{B} \vDash \text{the } t\text{th digit of } x \text{ is } 1\}.$$

Further, we can assume that the equality is definable in $\mathfrak{A}$ (otherwise we factorize $\mathfrak{A}$ by the relation of indistinguishability). Then in $\mathfrak{A} \dot{\times} \mathfrak{B}$ we have the relation

$$(a, b) =_A (c, d) \Leftrightarrow_{\text{df}} a = c,$$

which is the cylindrification of the equality of $\mathfrak{A}$. In $\mathfrak{A} \dot{\times} \mathfrak{B}$ we can also define the unary relation W,

$$W(x) \Leftrightarrow_{\text{df}} \bigcap_{t \in x^*} [t]_A \neq \varnothing,$$

by the formula $\exists z(\forall t \in x^*)\exists s(s \cong t \,\&\, s =_A z)$, in which $t \in x^*$ should be replaced by its definition. Now we extend the language of T by a unary predicate and we extend the interpretation of T by interpreting the unary predicate by W. (We shall denote this predicate also by W.) Let T' be the (complete) theory consisting of all the sentences which are true in $\mathfrak{A} \dot{\times} \mathfrak{B}$ in this interpretation. It is easy to check that $\cong$ is a congruence for W, i.e. the axioms of equality hold for W if the equality is interpreted by $\cong$. Therefore T' is also sequential.

Let $\tau(n)$ be a tree defined in T' by the following formula:

$$W(n) \tag{4.1}$$

$$\& \,\forall m, r, s\big(|m| \leqslant r < s \,\&\, m + 2^s \sqsubseteq n \Rightarrow \neg W(m + 2^r + 2^s)\big). \tag{4.2}$$

Thus, for every element n of τ, n encodes a sequence of numbers such that the intersection of the projections of the blocks of these numbers on A is nonempty and the sequence is saturated with respect to this property; namely, it is not possible to insert a new member between two elements of any initial segment of this sequences. (Recall that $m < 2^{|m|}$.)

Let J be the domain of τ. J need not be a cut. In case T' proves J is not a cut, let r_0 be the smallest number such that $r_0 \notin J$. Define $K(r)$ by

(4.3) there exists a $\sqsubseteq$-maximal m such that

$$|m| \leqslant r \,\&\, \tau(m) \,\&\, W(m + 2^{r_0}).$$

If $K(r_0)$ then some $n = m + 2^{r_0}$ would satisfy (4.1) and (4.2), thus r_0 would be in the domain of τ which is a contradiction. Hence we have $\neg K(r_0)$. Clearly for $r = 0$ we can take $m = 0$ in (4.3); if $K(r)$ and m is an element which satisfies (4.3), then either m or $m + 2^r$ is a maximal element needed for $r + 1$ in (4.3). Thus we can define in T' a cut I contained in J say by

$$I(r) \Leftrightarrow_{\text{df}} (\forall s \leqslant r) K(s).$$

Now we can replace τ by its restriction to I. Hence we can assume that T' proves that τ is a tree such that (4.1) and (4.2) hold for every element n of τ and the domain J of τ is a cut.

We shall apply Lemma 3.6 to τ. Let $\Phi(n)$ be a formula given by the lemma. First we suppose that the *second* part of the disjunction (3.20) holds, i.e. "Φ encodes an infinite set of successors of an element m of τ". Then we set

$$S = \left\{(a, b) \in A \times B \,\middle|\, \left[\mathfrak{A} \dot{\times} \mathfrak{B} \vDash \Phi((a, b))\right] \,\&\, a \in \bigcap_{t \in (a, b)^*} [t]_A \right\}. \tag{4.4}$$

S is definable in $\mathfrak{A} \dot{\times} \mathfrak{B}$ since (4.4) is equivalent to

$$\mathfrak{A} \dot{\times} \mathfrak{B} \models (\forall t \in (a, b)^*)\exists s(s \cong t \,\&\, s =_A (a, b)).$$

Condition (a) is satisfied because of (3.19) of Lemma 3.6. In order to prove (b), we have to show that, for $x, y \in S$, if $x \not\cong y$ then they differ in the first coordinate. Suppose x and y are of the following form:

$$x = m + 2^r < y = m + 2^s,$$

where $|m| \leqslant r < s$. By (4.2) (with $n = m + 2^s$) we have $\neg W(m + 2^r + 2^s)$, whence

$$\varnothing = \bigcap_{t \in (m+2^r+2^s)^*} [t]_A = \bigcap_{t \in x^*} [t]_A \cap \bigcap_{t \in y^*} [t]_A.$$

The first coordinate of x must be in $\bigcap_{t \in x^*}[t]_A$ while the first coordinate of y must be in $\bigcap_{t \in y^*}[t]_A$. Hence they cannot be equal.

Now suppose that the *first* part of the disjunction (3.20) holds, i.e. "Φ encodes an infinite branch of τ". We have to consider two cases.

Case 1. There exists m in Φ such that there is no smallest n with the property

$$\left[\mathfrak{A} \dot{\times} \mathfrak{B} \models \Phi(n) \,\&\, m \sqsubseteq n\right] \,\&\, \bigcap_{t \in m^*} [t]_A \neq \bigcap_{t \in n^*} [t]_A. \tag{4.5}$$

Let

$$G = \left\{ n \,\middle|\, \left[\mathfrak{A} \dot{\times} \mathfrak{B} \models \Phi(n) \,\&\, m \sqsubseteq n\right] \,\&\, \bigcap_{t \in m^*} [t]_A = \bigcap_{t \in n^*} [t]_A \right\}.$$

Then it is not difficult to find a formula which defines G in $\mathfrak{A} \dot{\times} \mathfrak{B}$. Since every element has a successor in $(\Phi, \leqslant)$ and $\leqslant$ coincides with $\sqsubseteq$ on Φ, $(G; \leqslant)$ is an ordering with successors too, and the same is true about

$$H = \left\{ r \,|\, (\exists n \in G)\left(\mathfrak{A} \dot{\times} \mathfrak{B} \models r = |n|\right)\right\}.$$

Choose $a \in \bigcap_{t \in m^*}[t]_A$ and define $S = H \cap (\{a\} \times B)$. Again it is easy to define S by a formula. Let us recall that $|n|$ is the largest element of n^*. Since

$$a \in \bigcap_{t \in m^*} [t]_A = \bigcap_{t \in n^*} [t]_A \subseteq \left[|n|\right]_A,$$

where $n \in G$, the set $\{a\} \times B$ meets every block $[r]$ for $r \in H$. Thus we have (a) of the lemma. Condition (b) is satisfied trivially for the second coordinate.

Case 2. For every m in Φ there exists a smallest n satisfying (4.5). Let

$$D = \{n \,|\, \text{for some } m \text{ in } \Phi, n \text{ is the smallest element satisfying (4.5)}\}.$$

Then $(D; \sqsubseteq)$ is an ordering with successors and the same is true about $(E; \leqslant)$, where

$$E = \left\{ r \,|\, (\exists m \in D)\left(\mathfrak{A} \dot{\times} \mathfrak{B} \models r = |m|\right)\right\}.$$

We define S by

$$S = \left\{ (a, b) \in A \times B \,\middle|\, (\exists n, m \in D)\left(n \text{ is the successor of } m \text{ in } (D, \sqsubseteq) \,\&\, (a, b) = |m| \,\&\, a \in \bigcap_{t \in m^*} [t]_A - \bigcap_{t \in n^*} [t]_A \right)\right\}.$$

S meets every block $[r]$ for $r \in E$, since, for every $m, n \in D$, n the successor of m,

$$[|m|]_A \supset \bigcap_{t \in m^*} [t]_A - \bigcap_{t \in n^*} [t]_A \neq \varnothing .$$

Hence (a) is satisfied. We shall show that if $r, s \in S$, $r \not\cong s$, then they differ in the first coordinate, thus proving condition (b). Let $r \cong |m|$, $s \cong |p|$, $m, p \in D$ and suppose $m < p$. Let n be the successor of m in $(D, \sqsubseteq)$. Then the first coordinate of r is in

$$\bigcap_{t \in m^*} [t]_A - \bigcap_{t \in n^*} [t]_A \subseteq \bigcap_{t \in m^*} [t]_A - \bigcap_{t \in p^*} [t]_A$$

(since $n^* \subseteq p^*$), while the first coordinate of s is in $\bigcap_{t \in p^*} [t]_A$. □

THEOREM 4.2. *Every sequential theory is connected.*

PROOF. Let T_0 be a sequential theory and let T be an arbitrary finite sequential fragment of it. Let T be interpreted in $\mathrm{Th}(\mathfrak{A} \dot{\times} \mathfrak{B})$. We shall show that T is interpretable either in $\mathrm{Th}(\mathfrak{A})$ or in $\mathrm{Th}(\mathfrak{B})$. Take S given by Lemma 4.1 for T, and suppose, for definiteness, that condition (b) is satisfied for the first coordinate. Extend T to the theory T' that contains a new predicate P, axioms of equality for it, and an axiom saying that each element in $(P; \leqslant)$ possesses a successor. Extend the interpretation of T to an interpretation of T' by interpreting $P(x)$ as $\exists y(S(y) \,\&\, x \cong y)$. Since T' is sequential, T has an interpretation in T' with a domain contained in P (by Theorem 3.5). This gives us an interpretation in $\mathrm{Th}(\mathfrak{A} \dot{\times} \mathfrak{B})$ with domain contained in S. The property (b) of S assures that the structure defined on S is preserved by the projection on the set A. By Lemma 1.2 this projection is definable in $\mathrm{Th}(\mathfrak{A})$, whence we have an interpretation of T in $\mathrm{Th}(\mathfrak{A})$. By Theorem 1.1 it follows that T_0 is connected. □

COROLLARY 4.3. *PA, ZF, GB and* $\mathrm{Th}(\omega; +, \cdot)$ *are connected.* □

5. Remarks and problems.

5.1. The smallest nonzero prime element in the lattice of interpretability types of **[5]** is the type of theories with finite models but without one-element models. The next element about which we know that it is prime is the type of the theory of linear orders without maximal elements (see **[1]**). Further up there are the types of the sequential theories; the smallest one is the type of the elementary sequential theory, which is also the type of Q (see **[9]**) and of Peano Arithmetic with induction only for bounded formulae; the largest one is the type of $\mathrm{Th}(\omega; +, \cdot)$. The largest element of the lattice, namely the type of inconsistent theories, is also prime. We conjecture that there are more prime elements than those listed above.

Problem 1. Is the theory of $(\omega; \leqslant)$ connected?

We think that a positive answer to Problem 1 might simplify the proof of Theorem 4.2. In fact, if one considers only one-dimensional interpretations then it is so.

5.2. It is well known that the lattice of one-dimensional interpretability types is different from that of **[5]** (e.g. the type of theories with finite but not one-element models splits into infinitely many types). Let us call a theory *one-connected* if its type in this lattice is prime. Since Theorem 1.1 holds also for one-dimensional interpretations (with "connected" and "interpretation" replaced by "one-connected" and "one-dimensional interpretation"), every sequential theory is one-connected. However there is a criterion for one-connectedness which is much easier to apply. If the supports A, B of structures $\mathfrak{A}$, $\mathfrak{B}$ are disjoint, we define the disjoint union $\mathfrak{A} \dot{\cup} \mathfrak{B}$ of $\mathfrak{A}$ and $\mathfrak{B}$ by $(A \cup B; R_1, \ldots, R_n, S_1, \ldots, S_m)$, where $R_1, \ldots, R_n$ and $S_1, \ldots, S_m$ are the relations of $\mathfrak{A}$ and $\mathfrak{B}$. Then one can prove

THEOREM 5.1. *A theory T is one-connected iff, for every two structures $\mathfrak{A}$ and $\mathfrak{B}$ with disjoint supports and every finite fragment T' of T if T' has a one-dimensional interpretation in* Th($\mathfrak{A} \dot{\cup} \mathfrak{B}$), *then T' has a one-dimensional interpretation in* Th($\mathfrak{A}$) *or in* Th($\mathfrak{B}$). □

Using this theorem one can prove an analog of Lemma 4.1. without use of trees. Thus the theorem that every sequential theory is one-connected has a simpler proof. Moreover, it is possible to prove that the theory of $(\omega; \leqslant)$ and the theory of a dense ordering are one-connected.

5.3. If a nonzero compact element t is prime, then there is a largest element among the elements smaller than t. (Recall that t is compact if and only if it contains a finitely axiomatizable theory.) The proof of Theorem 4.2 gives us no hint how this uniquely determined element looks.

Problem 2. What is the largest element below the type of Q or below the type of GB?

We only know that it is not ZF.

5.4. It is important that equality is treated as a nonlogical symbol. Let T_1 and T_2 be some incomparable sequential theories (in **[4]** incomparable extensions of PA are constructed). Suppose that equality is the unique common nonlogical symbol of T_1 and T_2. Then $T_1 \cup T_2$ is sequential, since it is an extension of a sequential theory (say T_1) in the logic with equality (see a remark at the beginning of §2). Thus the type of $T_1 \cup T_2$ is prime and therefore cannot be equal to the join of the types of T_1 and T_2.

This also shows that the types of extensions of PA do not form a sublattice. Hence the lattice investigated by Lindström **[3]** and Švejdar **[8]** is not embedded into Mycielski's lattice.

5.5. In **[10]** R. Vaught considered the question of which recursively axiomatizable theories can be axiomatized by a scheme. (E.g. PA is axiomatized by a scheme; for an exact definition see **[10]**.) He used a concept very similar to the concept of a sequential theory. He did not use any name for it thus we shall call it simply a *Vaught theory*. Let R be the theory with the same language as Q and with the following axiom schemes:

$$\underline{n} + \underline{p} = \underline{n+p}; \qquad \underline{n} \cdot \underline{p} = \underline{n \cdot p}; \qquad \underline{n} \neq \underline{p} \quad \text{for } n \neq p;$$
$$\forall x (x \leqslant \underline{n} \Rightarrow x = \underline{0} \vee \cdots \vee x = \underline{n}); \qquad \forall x (x \leqslant \underline{n} \vee \underline{n} \leqslant x),$$

where $\underline{k}$ denotes the kth numeral, cf. [9]. A Vaught theory is a recursively axiomatizable theory with equality which contains R relativized to a unary relation symbol N, a binary function symbol Val and the axioms

$$\forall t_1,\ldots,t_n \exists x(\mathrm{Val}(x,\underline{0}) = t_1 \,\&\, \cdots \,\&\, \mathrm{Val}(x,\underline{n}) = t_n), \tag{5.1}$$

where $n \in \omega$ and $\underline{k}$ denotes the kth numeral. He proved that every Vaught theory can be axiomatized by a schema plus the axioms (5.1). Clearly, $\mathrm{Val}(x, n) = t$ corresponds to $\beta(t, n, x)$ or to $x[n] = t$ in the notations of this paper. If we disregard the difference between having a function symbol and having only a formula for it, then every recursively axiomatizable sequential theory is a Vaught theory, since (5.1) follows from (2.2). Thus every recursively axiomatizable sequential theory can be axiomatized by a schema. On the other hand we have the following theorem.

THEOREM 5.2. *There exists a Vaught theory which is not connected (and hence not sequential).*

PROOF. Let T_1 and T_2 be some incomparable Vaught theories and suppose that they have disjoint languages. Then the type of $T_1 \cup T_2$ is not prime. We shall show that there is a Vaught theory T in this type. We define T to be $T_1 \cup T_2$ plus a new symbol = (with the axioms of equality), plus R (with new relation and function symbols) relativized to a new unary predicate N, plus axioms (5.1) with a new symbol Val. Now T is Vaught and $T_1 \cup T_2$ is interpretable in T, hence it remains to show that T is interpetable in $T_1 \cup T_2$. We shall find a two-dimensional local interpretation. Let $=_i$ be the equality of T_i. Let $\mathrm{Val}_i(x, n)$ be the symbol for which we have (5.1.) in T_i, and let $\underline{n}_i$ denote the nth numeral in T_i, $i = 1,2$. We shall describe a two-dimensional interpretation by way of using pairs of variables. A symbol of T_i, say a binary predicate $R((x_1, x_2), (y_1, y_2))$ is interpreted by $R(x_i, y_i)$; the equality $(x_1, x_2) = (y_1, y_2)$ by $x_1 =_1 y_1 \,\&\, x_2 =_2 y_2$, the numbers will be interpreted by the numbers of T_1 by way of fixing the second coordinate, thus $N((x_1, x_2))$ is interpreted by $x_2 =_2 \underline{0}_2 \,\&\, N_1(x_1)$, similarly $+$, $\cdot$, etc. If we want to interpret the first k axioms of (5.1), then we interpret $\mathrm{Val}((x_1, x_2),(n_1, n_2)) = (t_1, t_2)$ by

$$\begin{aligned}(n_1 =_1 \underline{0}_1 \,\&\, \mathrm{Val}_1(x_1,\underline{0}_1) =_1 t_1 \,\&\, \mathrm{Val}_2(x_2,\underline{0}_2) =_2 t_2)\\ \vee \cdots \vee (n_1 =_1 \underline{k}_1 \,\&\, \mathrm{Val}_1(x_1, \underline{k}_1) =_1 t_1 \,\&\, \mathrm{Val}_2(x_2, \underline{k}_2) = t_2).\end{aligned}$$

Then the axioms follow from the same axioms for Val_1 and Val_2.

Problem 3. Are there Vaught theories which are not one-connected?

5.6. Similarly as Vaught did in his paper we can introduce a set-theoretical version of the concept of a sequential theory. Namely, consider the following condition.

T contains equality (along with the axioms of equality) and it is possible to define a formula $\varepsilon(x, y)$ such that T proves

$$\exists x \forall t \neg\varepsilon(t, x); \qquad \forall x, y \exists z \forall t(\varepsilon(t, z) \Leftrightarrow \varepsilon(t, x) \vee t = y).$$

Every sequential theory satisfies this condition (interpret $\varepsilon(x, y)$ by

$$(\exists n \in N)\beta(x, n, y),$$

where N, $\leqslant$, β satisfy (2.1)–(2.6)), and every theory satisfying this condition is *almost* sequential; the difference is that instead of (2.1) we have only that $\leqslant$ is a linear *quasiordering* with successors. However, one can easily show that for every such theory there is a sequential theory with the same interpretability type.

5.7. Yet another definition was suggested by J. Krajíček. His idea is to index sequences not by numbers but by arbitrary objects. It turns out that his definition is equivalent to the version considered in 5.6.

REFERENCES

1. A. Ehrenfeucht and J. Mycielski, *Theorems and problems on the lattice of local interpretability*, manuscript.

2. H. Gaifman and C. Dimitracopoulos, *Fragments of Peano's arithmetic and MRDP theorem*, Monograph. No. 30, Enseign. Math., Logic and Algorithmic, Genève, 1982, pp. 187–206.

3. P. Lindström, *Some results on interpretability*, Proc. Fifth Scandinavian Logic Symposium, Aalborg Univ. Press, 1979, pp. 329–353.

4. R. Montague, *Theories incomparable with respect to relative interpretability*, J. Symbolic Logic **27** (1962), 195–211.

5. J. Mycielski, *A lattice of interpretability types of theories*, J. Symbolic Logic **42** (1977), 297–305.

6. J. B. Paris and C. Dimitracopoulos, *A note on undefinability of cuts*, J. Symbolic Logic (to appear).

7. ______, *Truth definitions for Δ_0 formulae*, Monograph. No. 30, Enseign. Math., Logic and Algorithmic, Genève, 1982, pp. 317–329.

8. V. Švejdar, *Degrees of interpretability*, Comment. Math. Univ. Carolin. **19** (1978), 789–813.

9. A. Tarski, A. Mostowski and R. M. Robinson, *Undecidable theories*, North-Holland, Amsterdam, 1953.

10. R. L. Vaught, *Axiomatizability by a schema*, Symbolic Logic **32** (1967), 471–479.

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